Assignment 1

Hand in no. 2, 4bc, 6 and 7 by September 12 .

1. A finite trigonometric series is of the form $a_0+\sum_{n=1}^N(a_n\cos nx+b_n\sin nx)$. A trigonometric polynomial is of the form $p(\cos x, \sin x)$ where $p(x,y)$ is a polynomial of two variables x, y.
   (a) Write down the general expressions for trigonometric polynomial of degree one, two and three.
   (b) Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.
2. Let f be a 2π-periodic function which is integrable over $[-\pi,\pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx = \int_J f(x)dx,$$
where I and J are intervals of length 2π.
3. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.
4. Here all functions are defined on $[-\pi,\pi]$. (a) Sketch their graphs as 2π-periodic functions, (b) find their Fourier series and (c) determine the convergence and uniform convergence of these Fourier series (if possible).
   (a)
$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx,$$
   (b)
$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x,$$
   (c)
$$f(x) = \begin{cases} 1, & x\in[0,\pi] \\ -1, & x\in[-\pi,0] \end{cases} \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x,$$
   (d)
$$g(x) = \begin{cases} x(\pi-x), & x\in[0,\pi] \\ x(\pi+x), & x\in(-\pi,0) \end{cases} \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x.$$
5. Let f be a 2π-periodic function which is infinitely many times differentiable on $\mathbb{R}$. Show that its Fourier coefficients are of order $o(1/n^k)$ for any $k\geq 1$, that is, $a_nn^k, b_nn^k\to 0$ as $n\to\infty$ for any k. Hint: Better use complex notation.
6. Let f be a 2π-periodic function whose derivative exists and is integrable on $[-\pi,\pi]$. Show that its Fourier series decay to 0 as $n\to\infty$ without appealing to Riemann-Lebesgue Lemma. Hint: Use integration by parts to relate the Fourier coefficients of f to those of f'.

7. Let f be a continuous 2π-periodic function. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebsegue lemma. Hint: Establish the formula

$$2a_n = \int_{-\pi}^{\pi} [f(y) - f(y + \pi/n)] \cos ny \, dy \ ,$$

using Problem 2.

8. Let f be a function on $[a, b]$ and $x \in [a, b]$. f is said to be locally Lipschitz continuous at x if there are some L and δ such that

$$|f(y) - f(x)| < L|y - x|, \quad |y - x| < \delta, \ y \in [a, b] \ .$$

Show that f is Lipschitz continuous at x whenever it is locally Lipschitz continuous at x.